Study of Fractional Differential Equations and its Applications

Submitted to

Swami Ramanand Teerth Marathwada University, Nanded

For the degree of

Doctor of Philosophy in Mathematics

Under

Faculty of Science

By

Ahmad Yousif Abdullah Salamooni

Yemen

Under the guidance of

Dr. D. D. Pawar

Director and Associate Professor,
School of Mathematical sciences,

S. R. T. M. U, Nanded

August - 2016

Study of Fractional Differential Equation and its Applications

Abstract:

we discuss standard approaches to the problem of fractional derivatives and fractional integrals (simply called differintegrals), namely the Riemann-Liouville, the Caputo and the sequential approaches. We prove the basic properties of differintegrals including the rules for their compositions and the conditions for the equivalence of various definitions. Further, we give a brief survey of the basic methods for the solving of linear fractional differential equations and mention limits of their usability. In particular, we formulate the theorem describing the structure of the initial-value problem for linear two-term equations. Finally, we consider some physical applications, in particular fractional advection- dispersion equation and the viscoelasticity problem. Dealing with the the first issue we derive the fundamental solution in the form of Levy α-stable distribution density and then we discuss relations between the generalized central limit theorem and the choice of the corresponding fractional model. In section concerning viscoelasticity we mention some typical models generalizing the standard ones via the replacement of the classical terms by the fractional terms. In particular, we derive the step response functions of those generalized systems.

Introduction:

The subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectlycontributed to its development. They included Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville. Ravi P. Agarwal, Mouffak Benchohra, And Samira Hamani.(2009). Studied The sufficient conditions are established for the existence of Solutions for a class of boundary value problems for fractional differential equations involving the Caputo fractional derivative. Ibtisam K. Hanan.(2011). used the homotopy analysismethod to solve special types of the initial value problems that consist of multi-fractional order integro differential equation in which the fractional derivative and fractional integral in them are described in the Caputo sense and Riemann-Liouville sense respectively. K. Malar, P. Karthikeyan and R. Arul.(2012). Studied the existence and uniqueness of solution for nonlocal Impulsive fractional Integro Differential equations involving the Caputo fractional derivative in a Banach Space. Anatoly A. Kilbas, Natalia V. Zhukovskaya , Studied the linear nonhomogeneous ordinary differential equations with three left-hand sided Lioville derivatives of fractional order. Rudolf Hilfer , Yury Luchko , Zivorad Tomovski . (2009) , used the operational calculus to solve the corresponding initial value problem for the general n-term linear equation with these generalized fractional derivatives of arbitrary order sand types with constant coefficients.Yang Xiao-Jun .(2010) . Studied the fractional trigonometric functions in complex-valued space and proposes a short outline of local fractional calculus of complex function in fractal spaces. Amera Almusharrf . (2011) .developed fractional trigonometry and generalize Wronskian determinant that can be used to determine the linear independence of a set of solutions to a system

of fractional differential equations. We also introduce a new set of fractional differential equations whose solutions are fractional trigonometric functions.

This synopsis "Study of Fractional Differential Equations and its Applications " based on the definitions of fractional derivative and integration , also on special functions as Gamma function , Beta function , Mittag-Leffler Function , and miller – Ross function.

Definition of the Riemann Liouville Fractional Integral:

The Riemann-Liouville fractional integral defined as

$$ {}_a D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt $$

Where $\alpha > 0$ is any non-negative real number, $f(x)$ be piecewise continuous on (a, x) and integrable on any finite subinterval of $[a, x]$.

Definition of the Riemann Liouville Fractional Derivative :

The fractional derivative can be defined using the definition of the fractional integral. suppose that on $n-1 < \alpha \le n$ and n is the smallest integer greater than on α .Then, the fractional derivative of $f(x)$ of order $\alpha > 0$, is

$$ {}_a D_x^{-\alpha} f(x) = \frac{d^n}{dx^n} [\, {}_a D_x^{-(n-\alpha)} f(x) \,] = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt $$

Definition of the Caputo's Fractional Derivative :

The Caputo's fractional derivative of $f(x)$ of order $\alpha > 0$, is

$${}_{a}^{C}D_{x}^{-\alpha}\, f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1}\, f^{(n)}(t)\, dt\,,\ \text{where}\ \ n-1<\alpha\le n\,.$$

Objectives Of The Research Work:

The synopsis will be divided into different chapters such as introduction of the work, solution of various fractional differential equations, existence and uniqueness theorem, Laplace fractional differential equations applications of fractional differential equations, result and discussion. further, each chapter will be subdivided into different sections if required .

Review of literature and development in the subject area:

1)Brief history:

The fractional calculus is a natural extension of the traditional calculus. It is like many other mathematical branches and ideas ; it has its origin in the pursuing for the extension of meaning. Well known examples are the extension of the integer numbers to the rational numbers, of the rational numbers to the real numbers, and of the real numbers to the complex numbers. The question ofextension of meaning in differential and integral calculus: Can the derivative $\frac{d^n y}{dx^n}$ of integer order $n>0$, be extended to n any order fractional, irrational or complex ? The answer of this question has led to the development of a new theory which is called fractional calculus . The concept of fractional calculus is believed to have emerged from a question raised in the year 1695 by Marquis de L.Hôpital. In a letter dated September 30th, 1695 L.Hôpital wrote to Leibniz asking him about a particular notation he had used in his publications for the n-th derivative $\frac{d^n f(x)}{dx^n}$ of the linear function $f(x)=x$. L.Hôpital

posed the question to Leibniz, what would be the result be if $n = 0$, Leibniz responded to the question, that " $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx:x}$ ".

2)Recent Works:

On the other side, many of researchers considers fractional differential equations, S. Z. Rida and A. A. M. Arafa developed a new application of the Mittag-Leffler Function method that will extend the application of the method to linear differential equations with fractional order. A new solution is constructed in power series. The fractional derivatives are described in the Caputo sense. To illustrate the reliability of the method, some examples are provided. The results reveal that the technique introduced here is very effective and convenient for solving linear differential equations of fractional order

Objective of research:

This thesis covers some analysis for the study of fractional-order advection-diffusion equations relevant to this anisotropic cardiac tissue model. The study of fractional-order equations is difficult: Firstly, fractional-order operators are nonlocal, i.e. the value of a fractional derivative of a function at a point in the domain depends on values of the function throughout the domain; and secondly,boundary conditions (traces) do not make sense in fractional Sobolev spaces of order $S \geq \frac{1}{2}$, so constraints must be defined on a region of non-zero

volume. We review and derive some relevant results on fractional Sobolev spaces, fractional-order operators and the nonlocal calculus developed by Du, Gunzburger, Lehoucq, Zhou (2011). We prove well-posedness of a general class of fractional-order elliptic problems.

Mothodology to be adopted:

The algorithms will be investigated. The solution of each problems that we set up in the introduction will proved or disproved with such counter examples.

Importance:

Applied mathematics has long provided a rich source of motivation for developments in Fractional differential equations Also, the fractional differential equations plays a central role in linear systems and control theory and dynamical systems.

Proposed work plan and applications:

1)Interdisplenary of application:

A number of intriguing questions remain unanswered about fractional differential equations.

2)Plan of research:

More than four scientific papers are expected to be published in international journals. The results will be presented at least at two international conferences with few dedicated workshops, and networking environment and opportunities to share

the scientific results with the European community. Participation to scientific workshops, meetings, specialized training and short-term research visits are expected during the PhD project.

References:

1) A.A. Kilbas, H. M. Srivastava,And J.J. Trujillo" Theory And Applications Of Fractional Differential equations" Amsterdam:Elsevier. 2006.

2) Ravi P. Agarwal, Mouffak Benchohra, And Samira Hamani. "Boundary Value Problems For Fractional DifferentialEquations" Georgian Mathematical Journal Volume 16 (2009), Number 3, 401,411.

3) Ibtisam K. Hanan. "The Homotopy Analysis Method for Solving Multi-Fractional Order Integro- Differential Equations" Journal of Al-Nahrain University. Vol.14 (3), September, 2011, pp.139-143.

4) Anatoly A. Kilbas, Natalia V. Zhukovskaya . "Solution Of Euler- Type Non-Homogeneous Differential Equations With Three Fractional Derivatives" Department of Mathematics and Mechanics Belarusian State University.

5) Rudolf Hilfer , Yury Luchko , ·Zivorad Tomovski "operational method for the solution of fractional differential equations with generalized Riemann liouville fractional derivatives"Diogenes Co. (Bulgaria). 2009.

6) Igor Podlubny," Fractional Differential Eqnations," San Diego: Academic Press . 1999 .

7) Adam Loverro " Fractional Calculus: History, Definitions and Applications for the Engineer " Department of Aerospace and Mechanical Engineering University of Notre Dame .2004.

8) Mehdi Dalir. Majid Bashour " Applications of Fractional Calculus"Applied Mathematical Sciences, Vol. 4, , no. 21, 1021 – 1032 . 2010 .

9) Shantanu Das." Functional Fractional Calculus " Springer-Verlag Berlin Heidelberg. 2011.

10) Kith B . Oldham , Jerome Spanier The Fractional Calculus : Theory and Application of Differentiation and Integration to Arbitrary Order"Academic Press,Inc .1974.

11) Amera Almusharrf ."Development Of Fractional Trigonometry And An Application Of Fractional Calculus To Pharmacokinetic Model " Western Kentucky University. 2011.

12) K. Malar, P. Karthikeyan and R. Arul."On The Existence Of Solutions Of Impulsive Fractional Integro-Differential Equations"nternational Journal Of Mathematics And ScientificComputing (Issn: 2231-5330), Vol. 2, No. 2, 2012.

13) M. Benchohra, J. Henderson and S. K. Ntouyas,"Impulsive Differential Equations and Inclusions", Hindawi Publishing Corporation, Vol. 2, New York, 2006.

14) S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional diffrential equations",Electron. J. Differential Equations, vol.2006, no. 36, pp. 1-12, 2006.

15) Zhi-Wei Lv, Jin Liang and Ti-Jun Xiao, "Solutions to Fractional Differential Equations with nonlocal initial condition in Banach Space", Advances in Difference Equations, vol. 2010, pp. 340-349, 2010.

16) A. Chauhan and J. Dabas, "Existence of mild solutions for impulsive fractional- order semilinear evoluation equations with nonlocal conditions", Electron. J. Differential Equations, vol. 2011, no. 107, pp. 1-10,2011.

17) B. Ahmad and S. Sivasundaram, "Some existence results for fractional integro-differetial equations with nonlinear conditions", Communications in Applied Analysis, vol. 12, no. 2,pp. 107-112, 2008.

18) S. Z. Rida and A. A. M. Arafa, "New Method for Solving Linear Fractional

Differential Equations", Department of Mathematics, Faculty of Science, South Valley University, Qena 83523, Egypt.

Signature

Research Scholar Research Supervisor